Oxford Cambridge and RSA

AS Level Further Mathematics A

Unit Y535/01 Additional Pure Mathematics

Tuesday 22 May 2018 – Afternoon

Time allowed: 1 hour 15 minutes

7066408982

You must have:
- Printed Answer Booklet
- Formulae AS Level Further Mathematics A

You may use:
- a scientific or graphical calculator

INSTRUCTIONS

- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- Do **not** write in the barcodes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.
- The acceleration due to gravity is denoted by $g\,\mathrm{m\,s^{-2}}$. Unless otherwise instructed, when a numerical value is needed, use $g = 9.8$.

INFORMATION

- The total mark for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- **You are reminded of the need for clear presentation in your answers.**
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **4** pages.

1 The points A, B and C have position vectors $6\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$, $13\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$ and $16\mathbf{i} + 6\mathbf{j} + 3\mathbf{k}$ respectively.

(i) Using the vector product, calculate the area of triangle ABC. **[5]**

(ii) Hence find, in simplest surd form, the perpendicular distance from C to the line through A and B. **[3]**

i. $\underline{b} - \underline{a} = \begin{pmatrix} 13 \\ 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 1 \end{pmatrix}$ $\quad \underline{c} - \underline{a} = \begin{pmatrix} 16 \\ 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 6 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \\ -1 \end{pmatrix}$

$\underline{c} - \underline{b} = \begin{pmatrix} 16 \\ 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 13 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$

$(\underline{b} - \underline{a}) \times (\underline{c} - \underline{a}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 7 & 0 & 1 \\ 10 & 4 & -1 \end{vmatrix} = \begin{pmatrix} -4 \\ 7+10 \\ 7\times4 \end{pmatrix} = \begin{pmatrix} -4 \\ 17 \\ 28 \end{pmatrix}$

$$\text{area } ABC = \frac{1}{2}\sqrt{4^2 + 17^2 + 28^2}$$

$$= \frac{33}{2}$$

ii. $AB = \sqrt{7^2 + 1^2} = \sqrt{50} = 5\sqrt{2}$

$$\text{area} = \frac{1}{2}(AB)d = \frac{33}{2} \Rightarrow d = \frac{33}{5\sqrt{2}} = \frac{33\sqrt{2}}{5\times2} = \frac{33\sqrt{2}}{10}$$

2 The surface with equation $z = 6x^3 + \frac{1}{9}y^2 + x^2y$ has two stationary points.

(i) Verify that one of these stationary points is at the origin. [4]

(ii) Find the coordinates of the second stationary point. [5]

i. $\left(\frac{dz}{dx}\right)_y = 18x^2 + 2xy \quad \left(\frac{dz}{dy}\right)_x = \frac{2}{9}y + x^2$

$@ (0,0), \ \frac{dz}{dx} \ \& \ \frac{dz}{dy} = 0$

$\therefore$ one stationary point is @ the origin

ii. $x, y \neq 0$

$18x^2 = -2xy \qquad \frac{2}{9}y = -x^2$

$y = -9x \qquad y = -\frac{9}{2}x^2$ $\Big\}$ $-9x = -\frac{9}{2}x^2 \Rightarrow x = 2$

$\because x \neq 0$

$y = -9(2) = -18$

$z = 6(-8) + \frac{1}{9}(18)^2 + (4 \times 18) = 12$

$S_2 = (2, -18, 12)$

3 Given that n is a positive integer, show that the numbers $(4n + 1)$ and $(6n + 1)$ are co-prime. **[3]**

$$a(4n+1) - b(6n+1) = n(4a-6b) + a - b$$

eliminate n: $a=3$, $b=2 \Rightarrow n(4(3)-6(2)) + 3-2$

$= 1$

let $h = HCF(4n+1, 6n+1)$

$\Rightarrow h \mid 1$

$\therefore h = 1$, $(4n+1)$ & $(6n+1)$ are co-prime

4 The group G consists of a set of six matrices under matrix multiplication. Two of the elements of G are

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}.$$

(i) Determine each of the following:

- $\mathbf{A}^2$
- $\mathbf{B}^2$ **[2]**

(ii) Determine all the elements of G. **[4]**

(iii) State the order of each non-identity element of G. **[3]**

(iv) State, with justification, whether G is

- abelian
- cyclic. **[2]**

i. $A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad B^2 = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1-1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad B^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

ii. $\underline{\mathbb{1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is an element of G $(A^2, B^2 \in G)$

$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ $\quad BA = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$

$(AB)^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = BA$

$ABA = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$ $\quad BAB = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$

$= ABA$

iii. A, B: order 2

AB, BA: Order 3

ABA (or BAB): Order 2

iv. G NOT abelian $\because AB \neq BA$

G NOT cyclic $\because$ no element of order 6, & also not abelian

5 For integers a and b, with $a \geqslant 0$ and $0 \leqslant b \leqslant 99$, the numbers M and N are such that

$$M = 100a + b \quad \text{and} \quad N = a - 9b.$$

(i) By considering the number $M + 2N$, show that $17 \mid M$ if and only if $17 \mid N$. **[4]**

(ii) Demonstrate step-by-step how an algorithm based on the result of part **(i)** can be used to show that $2058\,376\,813\,901$ is a multiple of 17. **[4]**

i. $M+2N = 102a - 17b = 17(6a-b)$

if $17|N \Rightarrow 17|M \Rightarrow 17|(17(6a-b)-2N)$

$\hookrightarrow$ both divisible by 17 so $17|M$

$\therefore 17|N \Rightarrow 17|M$

if $17|M \Rightarrow 17|2N \Rightarrow 17|(17(6a-b)-M)$

$\hookrightarrow$ both divisible by 17 & $HCF(2,17)=1$

$\therefore 17|M \Rightarrow 17|N$

$\therefore 17|M \Leftrightarrow 17|N$

ii. $2058376813901 \rightarrow 20583768139 - 9 = 20583768130$

$20583768130 \rightarrow 205837681 - 270 = 205837411$

$205837411 \rightarrow 2058374 - 99 = 2058275$

$2058275 \rightarrow 20582 - 675 = 19907$

$19907 \rightarrow 199 - 63 = 136$

$136 = 8 \times 17$

original number M is a multiple of 17

6 The Fibonacci sequence $\{F_n\}$ is defined by $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geqslant 2$.

(i) Show that $F_{n+5} = 5F_{n+1} + 3F_n$ [3]

(ii) Prove that F_n is a multiple of 5 when n is a multiple of 5. [5]

i. $F_n = F_{n-1} + F_{n-2} \quad \therefore \quad F_{n+5} = F_{n+4} + F_{n+3}$

$$= (F_{n+3} + F_{n+2}) + F_{n+3}$$
$$= 2F_{n+3} + F_{n+2}$$
$$= 2(F_{n+2} + F_{n+1}) + F_{n+2}$$
$$= 3F_{n+2} + 2F_{n+1}$$
$$= 3(F_{n+1} + F_n) + 2F_{n+1}$$
$$= 5F_{n+1} + 3F_n \quad \text{as required}$$

ii. $F_0 = 0$: divisible by 5

assume F_{5k} is divisible by 5

use result from (i) : $F_{5k+5} = 5F_{5k+1} + 3F_{5k}$

we have assumed F_{5k} is divisible by 5, & $5F_{5k+1}$ is also divisible by 5

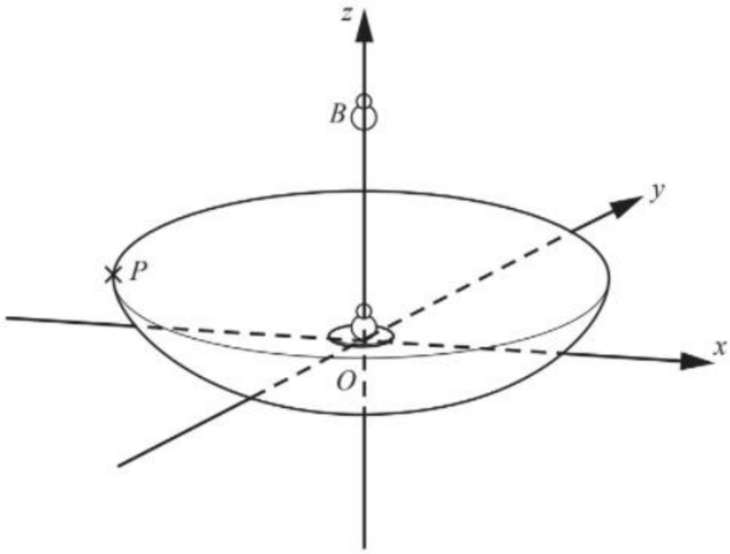
$\Rightarrow$ the sum of multiples of 5 is also divisible by 5

as this is true for $n=0$, & for $n=5(k+1)$ if true for $n=5k$, we conclude it is true for all +ve integers by induction

7 The 'parabolic' TV satellite dish in the diagram can be modelled by the surface generated by the rotation of part of a parabola around a vertical z-axis. The model is represented by part of the surface with equation $z = f(x, y)$ and O is on the surface.

The point P is on the rim of the dish and directly above the x-axis.

The object, B, modelled as a point on the z-axis is the receiving box which collects the TV signals reflected by the dish.

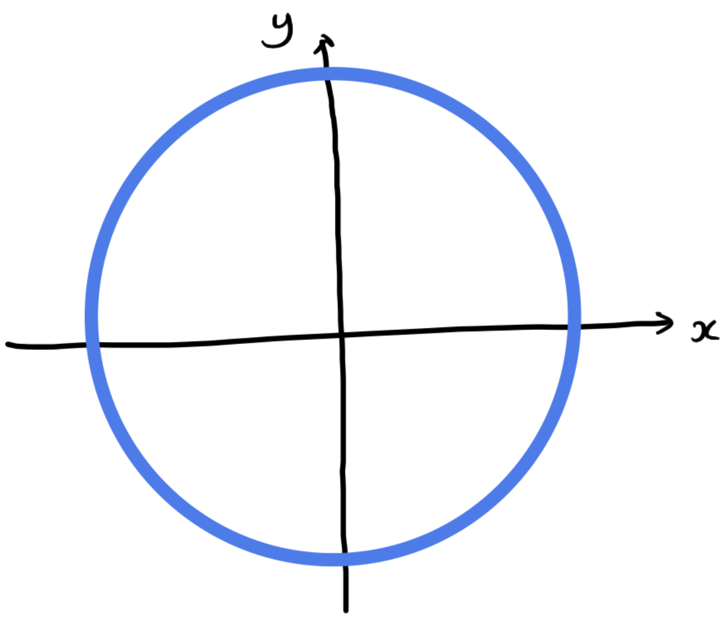


(i) The horizontal plane Π_1, containing the point P, intersects the surface of the model in a contour of the surface.

(a) Sketch this contour in the Printed Answer Booklet. **[1]**

(b) State a suitable equation for this contour. **[1]**

a)



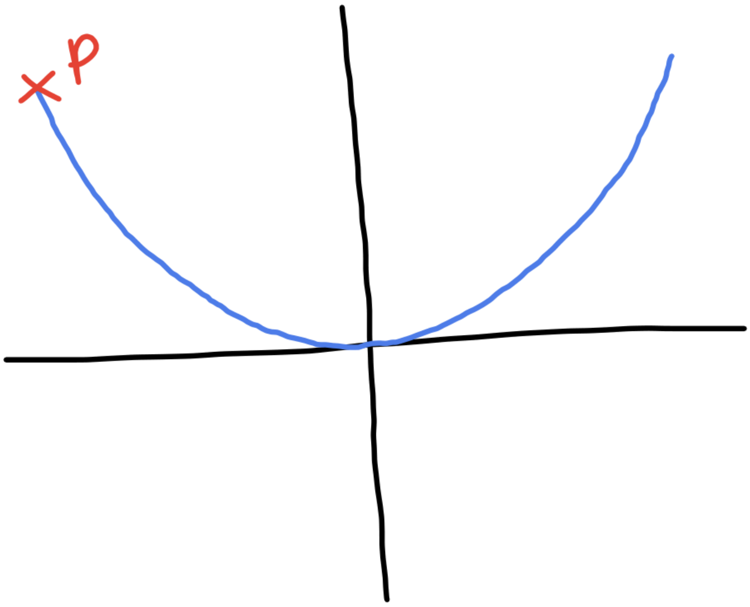
b) $f(x,y) = c$ or $z = c$

(ii) A second plane, Π_2, containing both P and the z-axis, intersects the surface of the model in a section of the surface.

(a) Sketch this section in the Printed Answer Booklet. **[1]**

(b) State a suitable equation for this section. **[1]**

a)



b) $z = kx^2$

$(z = f(x, 0))$

(iii) A proposed equation for the surface is $z = ax^2 + by^2$. What can you say about the constants a and b within this equation? Justify your answers. **[3]**

(iv) The real TV satellite dish has the following measurements (in metres): the height of P above O is 0.065 and the perimeter of the rim is 2.652. Using this information, calculate correct to three decimal places the values of

- a and b,
- any other constants stated within the answers to parts **(i)(b)** and **(ii)(b)**. **[4]**

iii. $z \geq 0 \Rightarrow a$ & b must be positive
surface generated by rotation so must be symmetrical in x & y
so by symmetry, $a = b$

iv. for (i)(b), $z = c$. height of $P = c$ so $c = 0.065$
perimeter $= 2\pi x = 2.652 \Rightarrow x$ coord of rim (which includes P) is 0.422

for (ii)(b), $k = a$

$$\Rightarrow a = \frac{z}{x^2} = \frac{0.065}{0.422^2} \text{ @ } P$$

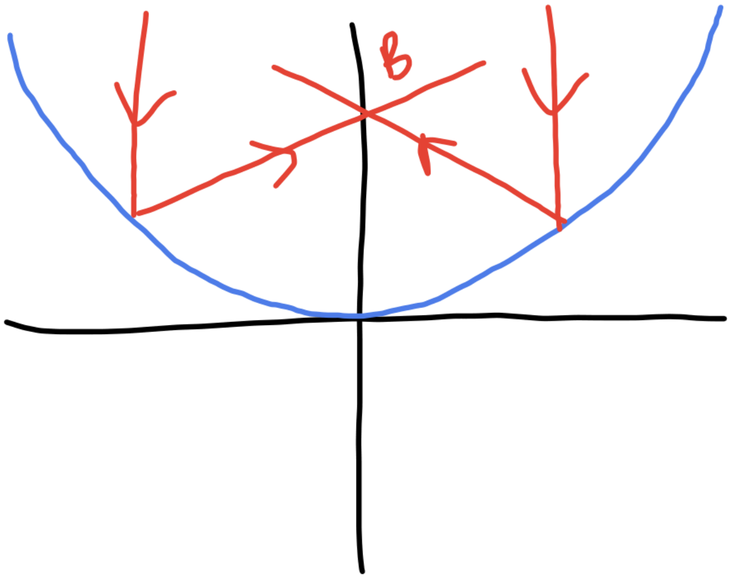
$a, b = 0.365$ (3s.f.)

(v) Incoming satellite signals arrive at the dish in linear "beams" travelling parallel to the z-axis. They are then 'bounced' off the dish to the receiving box at B.

- On the diagram for part **(ii)(a)** in the Printed Answer Booklet draw some of these beams and mark B.
- If the values of a and b were changed, what would happen? **[2]**

END OF QUESTION PAPER

V.



parabola changes shape so B must move to the new focus, otherwise the beams will miss